Exact Solution for the Stokes Problem of an Infinite Cylinder in a Fluid with Harmonic Boundary Conditions at Infinity

Andreas N. Vollmayr*, Jan-Moritz P. Franosch*, and J. Leo van Hemmen*

* Physik Department T35, TU München,
85747 Garching bei München, Germany
Correspondence and request for materials should be addressed to
J. Leo van Hemmen (e-mail: LvH@tum.de).

Abstract

We present an exact solution for the time-dependent Stokes problem of an infinite cylinder of radius $r = a$ in a fluid with harmonic boundary conditions at infinity. This is a 3-dimensional problem but, because of translational invariance along the axis of the cylinder it effectively reduces to a 2-dimensional one. The Stokes problem being a linear reduction of the full Navier-Stokes equations, we show how to satisfy the no-slip boundary condition at the cylinder surface and the harmonic boundary condition at infinity, exhibit the full velocity field for radius $r > a$, and discuss the nature of the solutions for the specific case of air at sea level.

1 Introduction: Navier-Stokes Equations

We denote a position in space by $\boldsymbol{x}$ and time by t. We describe an arbitrary fluid flow by its local velocity $\boldsymbol{v}(\boldsymbol{x},t)$, density $\varrho(\boldsymbol{x},t)$, and pressure $p(\boldsymbol{x},t)$ and denote the total derivative by

$$\frac{D\varrho(\boldsymbol{x},t)}{Dt} := \frac{\partial\varrho(\boldsymbol{x},t)}{\partial t} + \sum_{i=1}^{3} v_i(\boldsymbol{x},t)\frac{\partial\varrho(\boldsymbol{x},t)}{\partial x_i}\,. \tag{1}$$

For the sake of clarity, we will omit the position and time dependency and simply write $\varrho = \varrho(\boldsymbol{x},t)$. In an incompressible fluid of dynamic viscosity $\nu_0 = \mu/\varrho$ and with volume force $\boldsymbol{F}$ the Navier-Stokes equations are given by (Panton, 2005)

$$\frac{D\boldsymbol{v}}{Dt} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\frac{1}{\varrho_0}\,\mathrm{grad}\,p + \nu_0\triangle\boldsymbol{v} + \frac{1}{\varrho_0}\boldsymbol{F} \tag{2}$$

while incompressibility is explicitly taken care of by

$$\mathrm{div}\,\boldsymbol{v} = 0\,. \tag{3}$$

Finally, we will use the no-slip boundary condition, meaning that the velocity vanishes, i.e., $\boldsymbol{v} = \boldsymbol{0}$, at the surface of an infinite cylinder whose axis we take parallel to the z-axis. The cylinder radius equals $r = a$.

constant	value	unit
ν_0	$15.11 \cdot 10^{-6}$	m^2/s
ϱ_0	1.204	kg/m^3

Table 1: Measured values for the relevant material dependent constants for dry air at temperature 20°C and pressure $p_0 = 1013\,\mathrm{hPa}$.

The problem we are going to solve exactly is that of infinite cylinder in the above described fluid with harmonic boundary conditions at infinity. Since the cylinder itself is infinitely long we can simplify the problem to a 2-dimensional one. This we assume throughout what follows.

2 Solution of the Stokes Equations for an Infinite Cylinder in a Viscous Harmonic Flow

Stokes (1851) has already calculated the effect of internal friction of a harmonically moving fluid with angular frequency ω on the motion of pendulums with radius a in an approximation that is valid for $a\sqrt{\omega/\nu_0} \ll 1$. In the following we will give an analytical solution for the velocity field that is valid for all $r > a$ and a range of frequencies ω with Reynolds number $Re \ll 1$.

In practical work the stream velocity may well vary arbitrarily in time but, to simplify the problem, we solve it for a *harmonic* flow field. Hence the stream velocity at infinite distance from the cylinder is

$$\boldsymbol{v}_{\pm\infty} = v_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\omega t}\,, \tag{4}$$

as shown in Fig. 1.

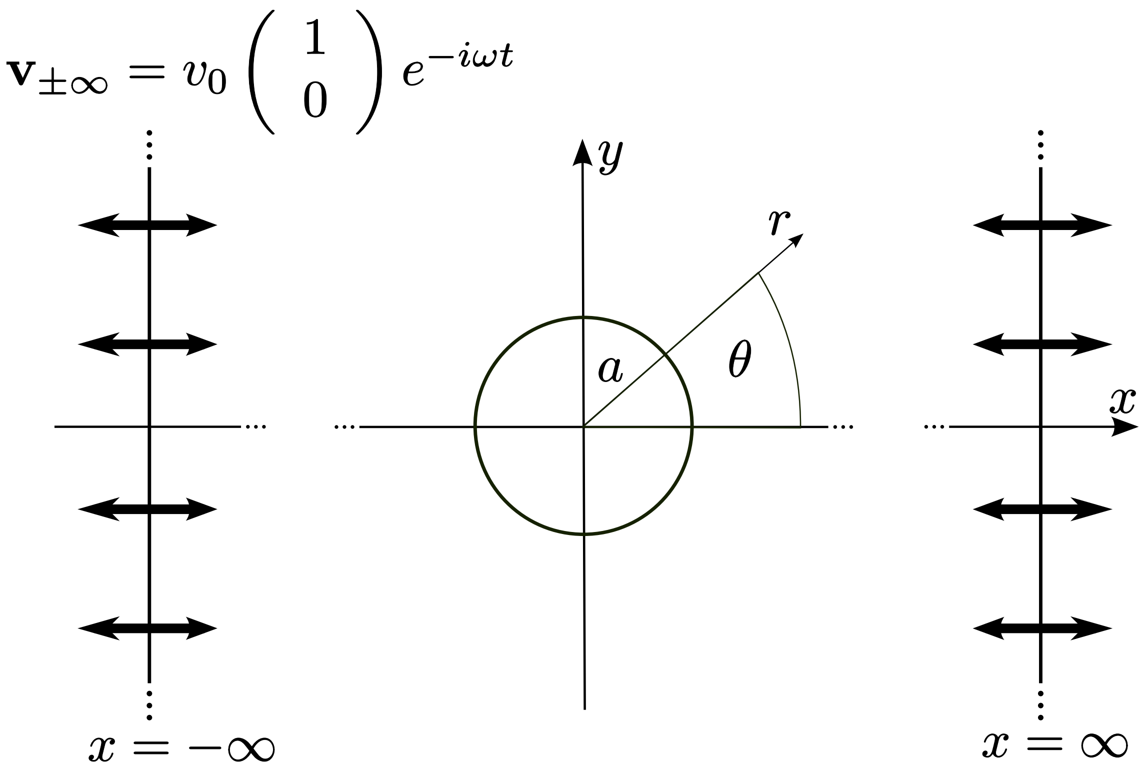

Figure 1: Cross-section through a cylinder with radius a in a flow field that is harmonic at infinite distance from the cylinder. Angle θ and distance r indicate the polar coordinates used here.

For $Re \ll 1$ the non-linear term $(\boldsymbol{v} \cdot \nabla)\,\boldsymbol{v}$ is negligible and we can use the *time-dependent Stokes equation*

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{\operatorname{grad} p}{\varrho_0} + \nu_0 \triangle \boldsymbol{v}\,. \tag{5}$$

It is *linear* in $\boldsymbol{v}$. Constant volume forces such as gravity do not play any role here since they only cause additional gravitational pressure and do not change the form of the equations as they can be incorporated into $\operatorname{grad} p$.

2.1 Boundary Conditions

The boundary condition (4) at infinite distance from the cylinder is consistent with to the time-dependent Stokes equation (5) as well as with the Navier-Stokes equations (2). This is important because at some distance from the cylinder the time-dependent Stokes equation is not valid any more.

On the cylinder surface we have no-slip boundary conditions,

$$\boldsymbol{v}(a,t) = 0\,. \tag{6}$$

Using the boundary condition (4) at infinity we can calculate the pressure at infinite distance from the origin. As the velocity at infinity is homogeneous

$$\triangle\, \boldsymbol{v}_\infty = 0 \tag{7}$$

and $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$ has been dropped we get, using (2),

$$\operatorname{grad} p_\infty = -\varrho_0 \frac{\partial \boldsymbol{v}_\infty}{\partial t}\,. \tag{8}$$

In view of Fig. 1 the boundary condition for the pressure at infinity is therefore

$$p_\infty = i x \varrho_0 \omega v_0 e^{-i\omega t}\,. \tag{9}$$

2.2 Solution of the Stokes Equation

Applying the divergence to both sides of the time-dependent Stokes equation (5) and taking advantage of we get

$$\frac{\partial \mathrm{div}\, \boldsymbol{v}}{\partial t} = -\frac{\mathrm{div\ grad}\, p}{\varrho_0} + \nu \triangle \mathrm{div}\, \boldsymbol{v} \tag{10}$$

and as $\mathrm{div}\, \boldsymbol{v} = 0$ we therefore find for the pressure

$$\triangle\, p = 0\,. \tag{11}$$

We first give a general solution to (11) that agrees with the boundary condition (9) and the symmetry of the problem.

In cylindrical coordinates, the boundary condition (9) for the pressure at infinity is

$$p(\infty,\theta) = i r \cos\theta \varrho_0 \omega v_0 e^{-i\omega t} \tag{12}$$

and (11) becomes

$$\triangle\, p = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2} = 0\,. \tag{13}$$

We give a general solution using the ansatz

$$p(r,\theta,t) = R(r)\, P(\theta)\, T(t)\ . \tag{14}$$

We get from (13), as $R(r)$ and $P(\theta)$ are functions depending on different variables, with a constant k independent of r and θ,

$$\frac{\partial^2 P}{\partial \theta^2} + k^2 P = 0 \tag{15}$$

and

$$\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2}+\frac{r}{R}\frac{\partial R}{\partial r}-k^2=0\,. \tag{16}$$

The general solution for $R(r)$ of (16) is

$$R(r)=\frac{A_{1,k}}{r^k}-A_{2,k}r^k \tag{17}$$

while for $P(\theta)$

$$P(\theta)=B_{1,k}\cos(k\theta)+B_{2,k}\sin(k\theta)\,. \tag{18}$$

The pressure must be continuous and hence

$$P(\theta)=P(\theta+2\pi) \tag{19}$$

so that $k=0,1,2,\dots$. As a general solution to (13) we get

$$p(r,\theta,t)=\sum_{k=0}^{\infty}\left[\frac{A_{1,k}}{r^k}-A_{2,k}r^k\right]\left[B_{1,k}\cos\left(k\theta\right)+B_{2,k}\sin\left(k\theta\right)\right]T(t)\,. \tag{20}$$

To comply with (12) the solution cannot contain any power of r^k with $k>1$ since the functions $\cos(k\theta)$ and $\sin(k\theta)$ are linearly independent for all $k=0,1,2,\dots$. As the Stokes equation is linear we set $T(t)=\exp(-i\omega t)$. Expressions with $k=0$ are irrelevant since they only result in an additive constant to the pressure. Possible solutions are therefore

$$p(r,\theta,t)=\left\{\sum_{k=1}^{\infty}\frac{1}{r^k}\left[C_{1,k}\cos\left(k\theta\right)+C_{2,k}\sin\left(k\theta\right)\right]+ir\varrho_0\omega v_0\cos\theta\right\}e^{-i\omega t}\,. \tag{21}$$

The problem is symmetric with respect to the x-axis and therefore

$$p(r,\theta,t)=p(r,-\theta,t) \tag{22}$$

so that solutions depending on $\sin(k\theta)$ must be zero. As we get another solution of the time-dependent Stokes equations by applying the transformation $\boldsymbol{v}\to-\boldsymbol{v}$ and $p\to-p$, which in our case is equivalent to mirroring with respect to the y-axis, it holds that $p(-x,y)=-p(x,y)$ or, in polar coordinates,

$$p(r,\theta,t)=-p(r,\pi-\theta,t)\ . \tag{23}$$

Since

$$\cos[k\left(\pi-\theta\right)]=\begin{cases}\cos(k\theta) & \text{for } k \text{ even}\\ -\cos(k\theta) & \text{for } k \text{ odd,}\end{cases} \tag{24}$$

the solutions depending on $\cos(k\theta)$ with even k must vanish. The remaining most general solution that complies with the boundary conditions is

$$p(r,\theta,t)=\left[\sum_{k=1}^{\infty}\frac{1}{r^k}C_k\cos\left(k\theta\right)+ir\varrho_0\omega v_0\cos\theta\right]e^{-i\omega t}\,. \tag{25}$$

We do not know yet the boundary condition for the pressure at the surface of the cylinder as it results from the flow field. To simplify further calculations we use the simple ansatz

$$p(r,\theta,t) = \left[\frac{C}{r} + ir\varrho_0\omega v_0\right]\cos\theta e^{-i\omega t} \tag{26}$$

and verify that it is consistent with the Stokes equations (5).

Since we are in two dimensions, the incompressibility condition reads in polar coordinates

$$\operatorname{div}\boldsymbol{v} = \frac{1}{r}\frac{\partial\left(rv_r\right)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} = 0 \tag{27}$$

and the time-dependent Stokes equations are given by

$$\frac{\partial v_r}{\partial t} = -\frac{1}{\varrho_0}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2}\right], \tag{28}$$

$$\frac{\partial v_\theta}{\partial t} = -\frac{1}{r\varrho_0}\frac{\partial p}{\partial\theta} + \nu\left[\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r^2}\right]. \tag{29}$$

We now substitute the pressure ansatz (26). With the help of (27) the above equations decouple so as to give an equation for v_r alone,

$$\frac{\partial v_r}{\partial t} = \left[\frac{C}{\varrho_0 r^2} - i\omega v_0\right]\cos\theta e^{-i\omega t} + \nu\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{3}{r}\frac{\partial v_r}{\partial r} + \frac{v_r}{r^2}\right] \tag{30}$$

while

$$v_\theta = \int\frac{\partial\left(rv_r\right)}{\partial r}d\theta\,. \tag{31}$$

A solution of the inhomogeneous equation (30) is

$$v_r(r,\theta,t) = \left[v_0 + \frac{AJ_1(j^+\beta r)}{\varrho_0\omega r} + \frac{BK_1(j^-\beta r)}{\varrho_0\omega r} + \frac{iC}{\varrho_0\omega r^2}\right]\cos\theta e^{-i\omega t} \tag{32}$$

with Bessel function J_1 of first order and second kind, modified Bessel function K_1 of first order and second kind, and

$$j^\pm := \frac{1}{\sqrt{2}}\left(1\pm i\right)\,,\ \beta := \sqrt{\frac{\omega}{\nu_0}}\,. \tag{33}$$

The constants A, B and C are still to be determined. The general solution of (30) is the sum of the general solution of the homogeneous equation plus a special solution of the inhomogeneous equation. If, however, the boundary conditions can be satisfied by the inhomogeneous solution as given above we need not care about the homogeneous solution any more.

The Bessel function of the second kind $J_1(j^+\beta r)$ with complex argument

$$j^+\beta r = \frac{1}{\sqrt{2}}\left(1+i\right)\sqrt{\frac{\omega}{\nu_0}}r \tag{34}$$

diverges to infinity as $r \to \infty$. Since the velocity has to be finite at infinity we must set $A = 0$. At the cylinder surface we have the no-slip boundary condition

$$v_r(a,\theta,t) = \left[v_0 + \frac{BK_1(j^-\beta a)}{\varrho_0\omega a} + \frac{iC}{\varrho_0\omega a^2}\right]\cos\theta e^{-i\omega t} = 0 \tag{35}$$

and we therefore obtain

$$B = -\frac{\varrho_0 a^2\omega v_0 + iC}{aK_1(j^-\beta a)}\,. \tag{36}$$

Substituting the constant B into (32) we get

$$v_r(r,\theta,t) = \left\{v_0 + \frac{iC}{\varrho_0\omega r^2} - \frac{a}{r}\left(v_0 + \frac{iC}{\varrho_0\omega a^2}\right)\frac{K_1(j^-\beta r)}{K_1(j^-\beta a)}\right\}\cos\theta e^{-i\omega t} \tag{37}$$

while (31) and a little algebra provide us with

$$\begin{aligned} v_\theta(r,\theta,t) = &-\left[v_0 - \frac{iC}{\varrho_0\omega r^2} + \frac{a}{r}\left(v_0 + \frac{iC}{\varrho_0\omega a^2}\right)\frac{K_1(j^-\beta r)}{K_1(j^-\beta a)} + \right. \\ &\left. + a\beta j^-\left(v_0 + \frac{iC}{\varrho_0\omega a^2}\right)\frac{K_0(j^-\beta r)}{K_1(j^-\beta a)}\right]\sin\theta e^{-i\omega t}\,. \end{aligned} \tag{38}$$

There is one last constant C that still has to be determined. We solve the no-slip boundary condition $v_\theta(a,\theta,t) = 0$ at the cylinder surface for C and find

$$C = i\varrho_0 a^2\omega v_0\left(1 + \frac{2j^+}{\beta a}\frac{K_1(j^-\beta a)}{K_0(j^-\beta a)}\right)\,. \tag{39}$$

As an abbreviation we introduce

$$f(r) := \frac{2j^+K_1(j^-\beta r)}{K_0(j^-\beta a)} \tag{40}$$

and with

$$\beta = \sqrt{\frac{\omega}{\nu_0}} \tag{41}$$

we finally obtain the full solution for $r > a$,

$$v_r(r,\theta,t) = v_0\cos(\theta)e^{-i\omega t}\left[1 - \frac{a^2}{r^2} + \frac{f(r)}{\beta r} - \frac{af(a)}{\beta r^2}\right]\,, \tag{42}$$

$$v_\theta(r,\theta,t) = v_0\sin(\theta)e^{-i\omega t}\left[-1 - \frac{a^2}{r^2} + 2\frac{K_0(j^-\beta r)}{K_0(j^-\beta a)} + \frac{f(r)}{\beta r} - \frac{af(a)}{\beta r^2}\right]\,, \tag{43}$$

$$p(r,\theta,t) = iv_0\varrho_0\omega\cos(\theta)e^{-i\omega t}\left[r + \frac{a^2}{r} + \frac{af(a)}{\beta r}\right]\,. \tag{44}$$

Figure 2 shows the pressure distribution around the cylinder. Figures 3, 4, and 5 contain plots of the fluid velocity. The fluid velocity always reaches its undisturbed value v_0 for $r \to \infty$, as required by the boundary conditions at infinity. For high frequencies, the fluid velocity reaches its undisturbed value for much shorter distances than for lower frequencies.

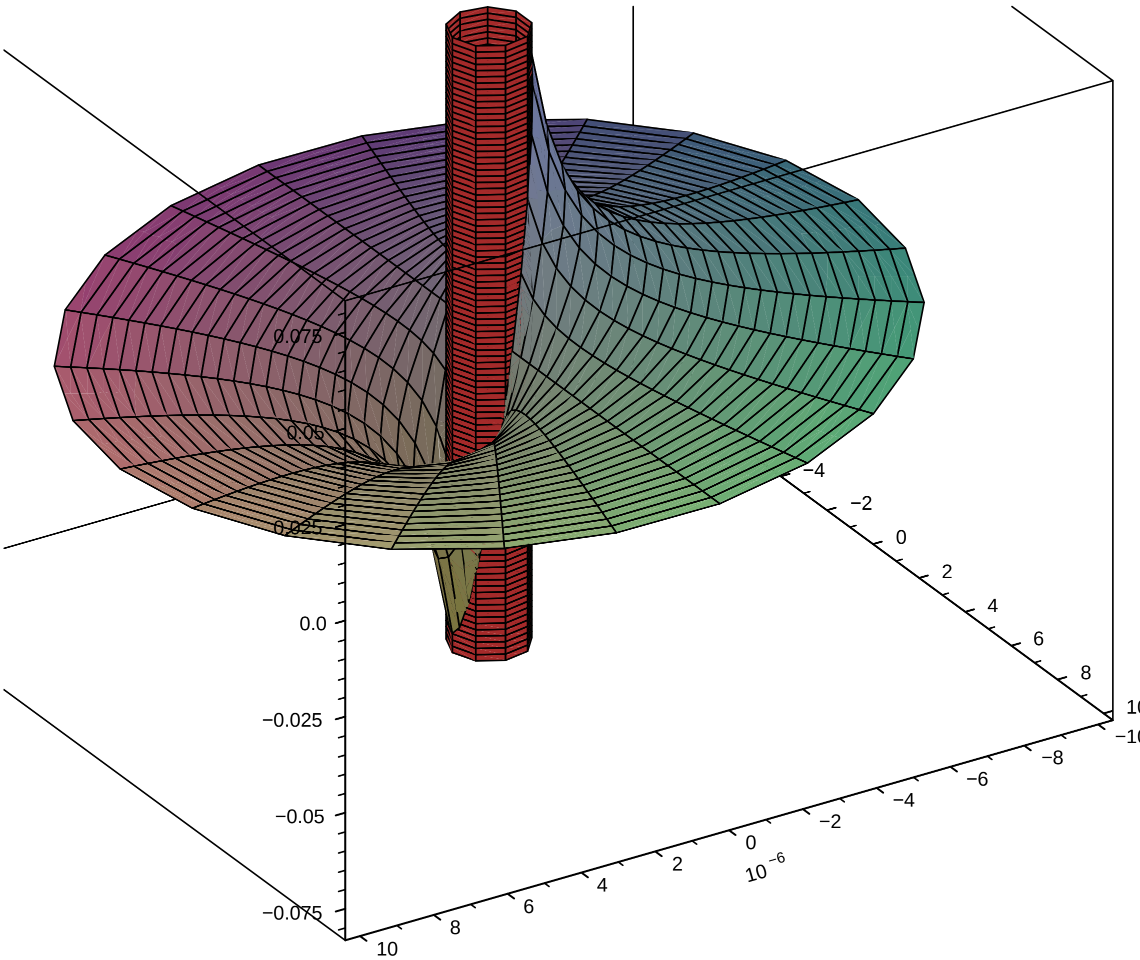

Figure 2: Pressure distribution $\Re[p(r, \theta, t = 0)]$ around a cylinder (red) with radius $a = 1\,\mu$m for frequency $f = \omega/2\pi = 10\,$Hz in arbitrary units. The pressure decreases steeply within a few cylinder diameters. We assume dry air in all figures.

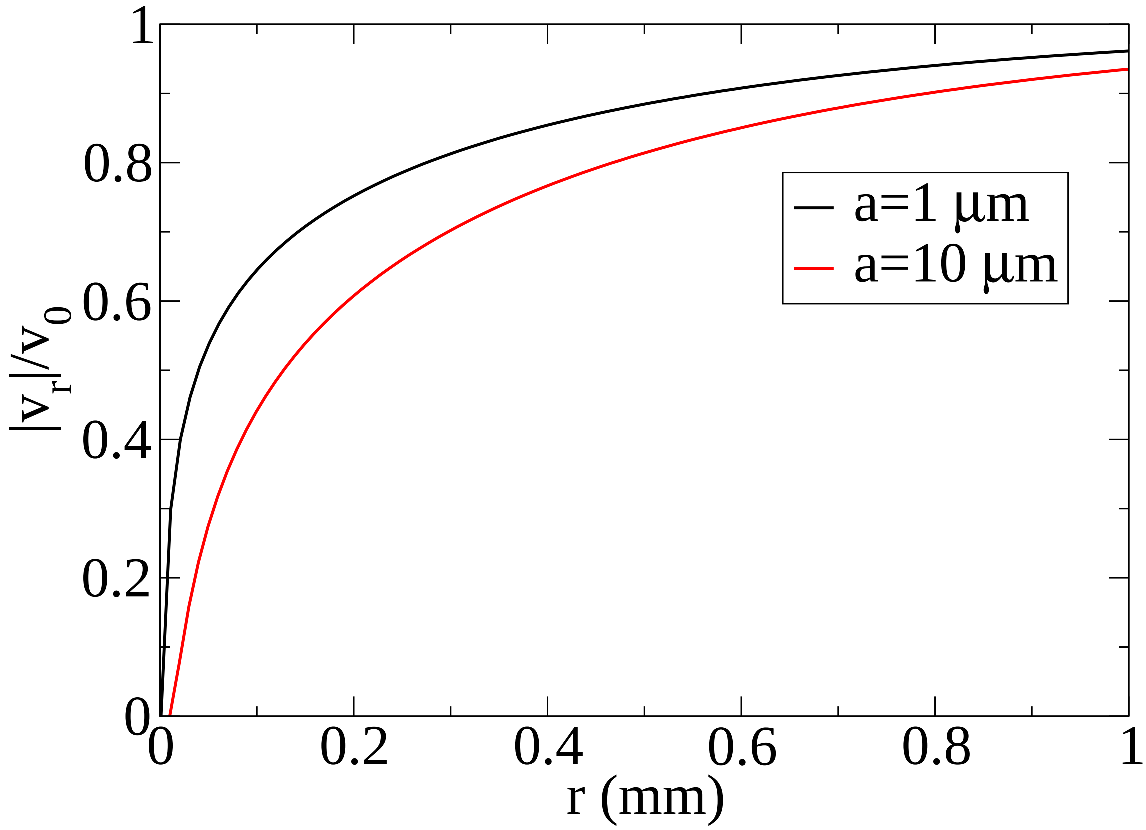

Figure 3: Amplitude of radial relative velocity $|v_r(r, \theta = 0)|/v_0$ of (42) in dependence upon the distance r to a cylinder with radius a for frequency $f = \omega/2\pi = 10\,\mathrm{Hz}$. The relative velocity approaches 1 for $r \to \infty$, as required by the boundary conditions. The thicker the cylinder, the farther the influence of the cylinder on the velocity field.

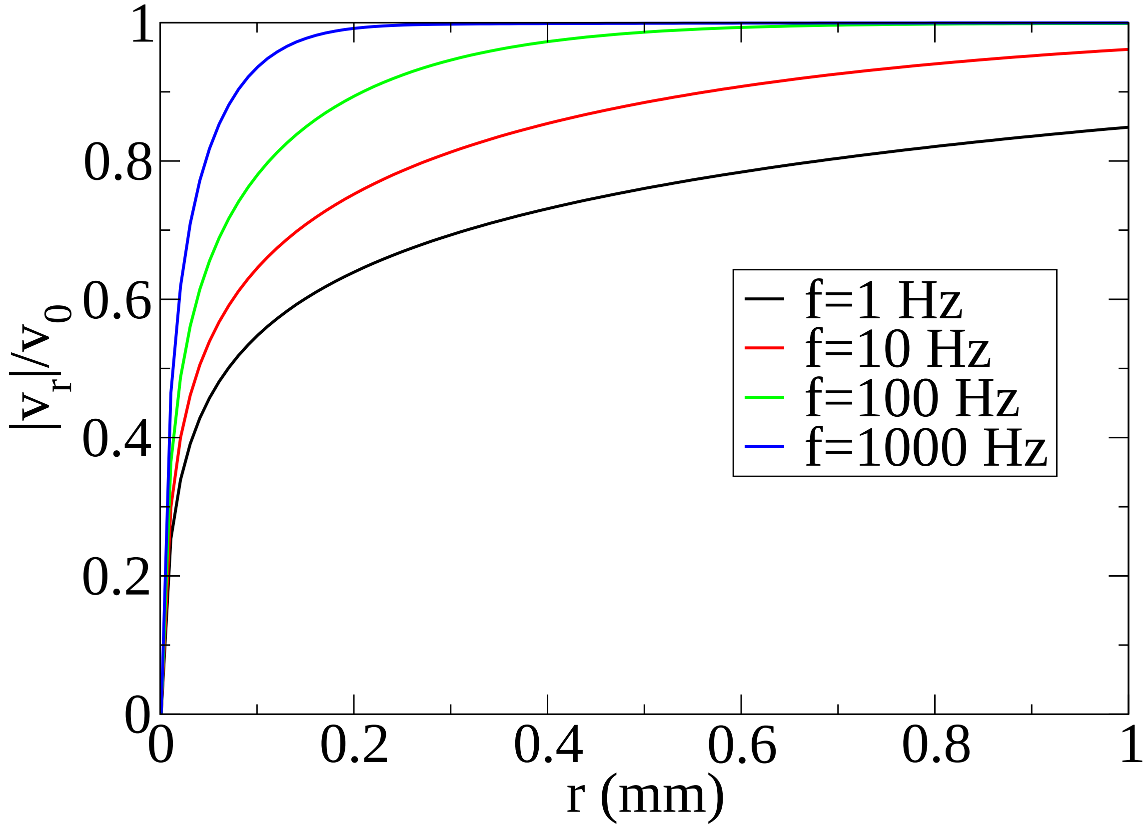

Figure 4: Amplitude of radial relative velocity $|v_r(r, \theta = 0)|/v_0$ of (42) in dependence upon the distance r to a cylinder with radius $a = 1\,\mu$m for different frequencies. For high frequencies, the fluid velocity reaches its undisturbed value at much shorter distances than for lower frequencies.

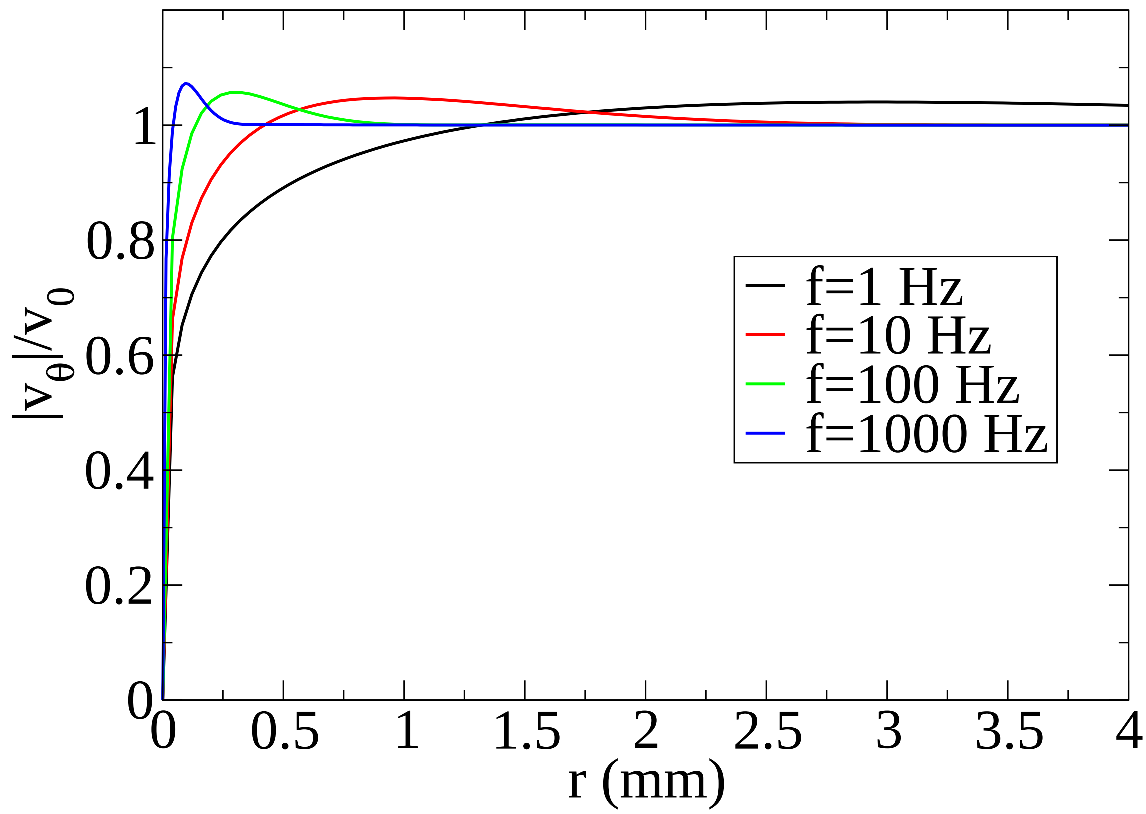

Figure 5: Amplitude of relative velocity $|v_\theta(r, \theta = \pi/2)|/v_0$ of (43) in dependence upon the distance r to a cylinder with radius $a = 1\,\mu$m for different frequencies. As the fluid must flow around the cylinder, relative velocities larger than 1 occur. For high frequencies, the fluid velocity reaches its undisturbed value at much shorter distances than for lower frequencies, as in Fig. 4.

3 Force on a Cylinder

The force $\boldsymbol{f}$ acting on a surface element is given by stress tensor $\mathbf{\Pi}$ operating on the normal vector $\boldsymbol{n}$ of the surface,

$$\boldsymbol{f} = \mathbf{\Pi}\boldsymbol{n}\,. \tag{45}$$

In polar coordinates the stress tensor is given by (Lamb, 1932, Art. 328a)

$$\mathbf{\Pi} = \begin{pmatrix} -p + 2\mu_0\partial_r v_r & \mu_0\left(\frac{1}{r}\partial_\theta v_r + \partial_r v_\theta - \frac{1}{r}v_\theta\right) \\ \mu_0\left(\frac{1}{r}\partial_\theta v_r + \partial_r v_\theta - \frac{1}{r}v_\theta\right) & -p + 2\mu_0\left(\frac{1}{r}\partial_\theta v_\theta + \frac{1}{r}v_r\right) \end{pmatrix}\,. \tag{46}$$

The normal vector on the cylinder surface reduced to a circle is

$$\boldsymbol{n} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}\,. \tag{47}$$

Because of (45) and (46), the force f_x per area in x-direction is given by

$$f_x = (-p + 2\mu_0\partial_r v_r)\cos\theta - \mu_0\left(\frac{1}{r}\partial_\theta v_r + \partial_r v_\theta - \frac{1}{r}v_\theta\right)\sin\theta\,. \tag{48}$$

The total force F_y in y-direction is zero because of symmetry. The total force F_x in x-direction per cylinder length is

$$F_x = \int_0^{2\pi}\left[(-p + 2\mu_0\partial_r v_r)\cos\theta - \mu_0\left(\frac{1}{r}\partial_\theta v_r + \partial_r v_\theta - \frac{1}{r}v_\theta\right)\sin\theta\right] a\,d\theta\,.$$

Substituting of the solution (42–44) and integration gives

$$F_x = -i2\pi\varrho_0 v_0 \omega a\left[a + \frac{f(a)}{\beta}\right]\; e^{-i\omega t}\,. \tag{49}$$

Figure 6 shows the force acting on a cylinder in dependence upon the fluid frequency. For low enough frequencies and thin cylinders forces that are independent of viscosity, so-called "buoyancy" forces,

$$F_p = -i2\pi\varrho_0 v_0 \omega a^2\; e^{-i\omega t}\,, \tag{50}$$

can be neglected as compared to viscous forces. When calculating the force on an oscillating cylinder with angular frequency ω additional buoyancy forces proportional to $\varrho_0 v_0 \omega a^2$ occur (Panton, 2005). If these are negligible, however, we need only consider the *relative* velocity v_0 between cylinder and fluid even in case of an oscillating cylinder so as to find

$$F_x \approx -i2\pi\varrho_0\, v_0\, \omega\, \frac{a f(a)}{\beta}\; e^{-i\omega t}\,. \tag{51}$$

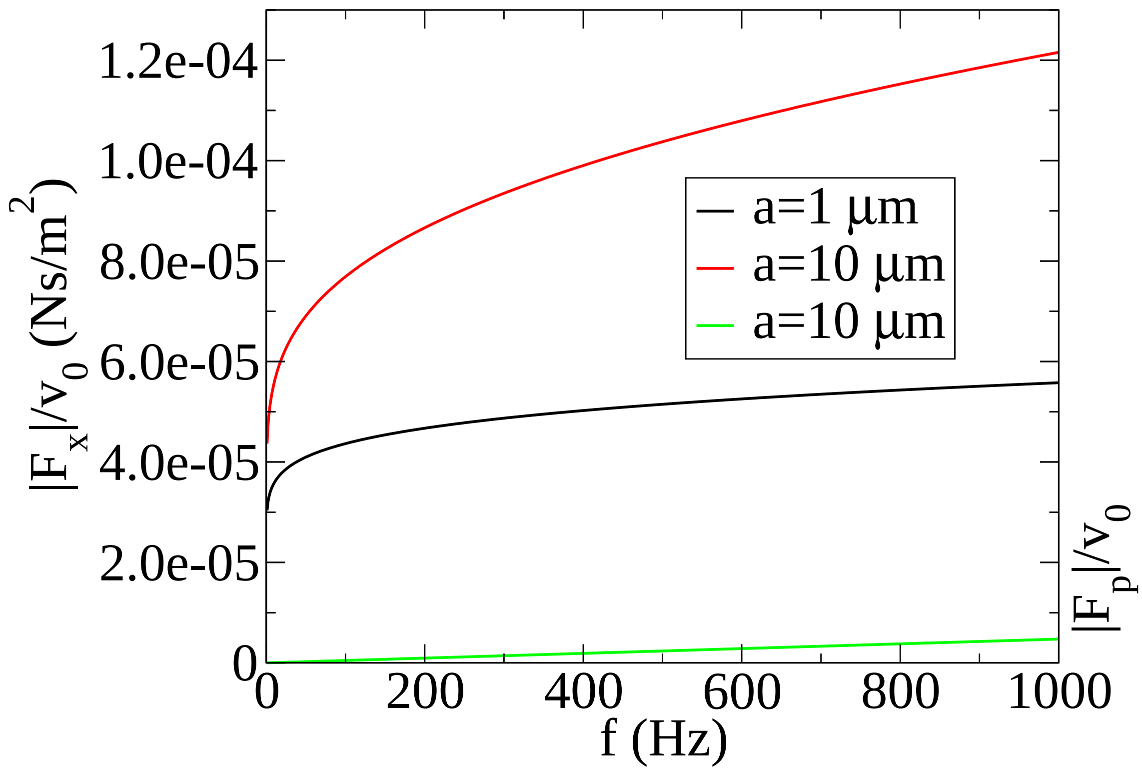

Figure 6: The amplitude of the force $|F_x|/v_0$ per cylinder length relative to fluid velocity v_0 acting upon a cylinder that stands still; cf. (49). The force is plotted for different cylinder radii a in dependence upon the frequency f. For a thin cylinder ($a = 1\,\mu$m) the force is more or less independent of the frequency in the range 100–1000 Hz. For thicker cylinders ($a = 10\,\mu$m) the force increases with frequency. The "buoyancy" force F_p (green graph) of (50) is negligible in comparison to viscous forces.

4 Discussion

For a very long cylinder in the low Reynolds number regime time-dependent incompressible Stokes equations hold. For harmonic flow, these can be solved analytically giving the flow and pressure field around a cylinder. However, the approximation of an infinitely long cylinder is only fulfilled when the stream velocity converges to its undisturbed constant value within a short range, so that the cylinder must be much longer than the distance where the velocity has reached about 90 % of the undisturbed value. This precondition is not fulfilled for very low frequencies as shown in Figs. 4 and 5.

The velocity field and viscous forces can be calculated analytically using the linear time-dependent Stokes equations as an approximation. In contrast to Stokes (1851), who computed the velocity field in the direct neighborhood of a cylinder, and, hence, the force for small radius a, we provide the *full velocity field* for *all* $r > a$.

References

Lamb, H. (1932). *Hydrodynamics*. Cambridge University Press, Cambridge, 6. edition.

Panton, R. L. (2005). *Incompressible Flow*. Wiley, Hoboken, 3. edition.

Stokes, G. G. (1851). On the effect of the internal friction of fluids. *Transactions of the Cambridge Philosophical Society*, **IX**:8.